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Gravitational Lens Time Delays and Gravitational Waves

Joshua A. Frieman[1,2], Diego D. Harari[3], and Gabriela C. Surpi[3]

[1]*NASA/Fermilab Astrophysics Center*

Fermi National Accelerator Laboratory

P.O. Box 500, Batavia, IL 60510, USA

[2]*Department of Astronomy and Astrophysics*

University of Chicago, Chicago, IL 60637

[3]*Departamento de Física, Facultad de Ciencias Exactas y Naturales*

Universidad de Buenos Aires

Ciudad Universitaria - Pab. 1, 1428 Buenos Aires, Argentina

Abstract

Using Fermat's principle, we analyze the effects of very long wavelength gravitational waves upon the images of a gravitationally lensed quasar. We show that the lens equation in the presence of gravity waves is equivalent to that of a lens with different alignment between source, deflector, and observer in the absence of gravity waves. Contrary to a recent claim, we conclude that measurements of time delays in gravitational lenses cannot serve as a method to detect or constrain a stochastic background of gravitational waves of cosmological wavelengths, because the wave-induced time delay is observationally indistinguishable from an intrinsic time delay due to the lens geometry.

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I. INTRODUCTION

A stochastic background of gravitational waves of cosmological wavelengths may arise in the early Universe, for instance as a consequence of quantum effects during a period of inflationary expansion, or as the result of gravitational radiation by oscillating cosmic strings. Its presence could be manifested as a large angular scale anisotropy in the cosmic microwave background, induced by the Sachs-Wolfe effect [1], the differential redshifting of photons in the presence of tensor metric perturbations. It is possible that a significant fraction of the anisotropy measured by the COBE DMR experiment [2] is due to cosmological gravitational waves. [3] So far, the microwave observations cannot determine how much of the anisotropy is due to tensor perturbations (gravitational waves) and how much to scalar (energy-density) fluctuations.

Another potential method to reveal the presence of gravitational waves of cosmological wavelengths was recently suggested by Allen, [4,5] namely, to use measured time delays between gravitationally lensed multiple images of distant quasars. Gravitationally lensed multiple images of a source such as a quasar arrive at the Earth at different times if the source, deflector (the lensing body), and observer are not in perfect alignment, because there is a difference in geometric path lengths between, and in the deflector's gravitational potential traversed by, the different light rays. We shall call these two effects the 'intrinsic' time delay of the lens. For a lens geometry where L is the distance between observer and deflector and 2η is the angular separation between the images, the typical intrinsic time delay is $\Delta T \sim L\eta^2$. Any 'extrinsic' perturbation to the spacetime metric (i.e., not associated with the lens itself) would be expected to cause additional time delays between the images. For example, an additional time delay would be induced by a cosmological background of very long wavelength gravitational waves. [4,5] As shown by Allen, for waves with frequency $\omega \sim L^{-1}$, the gravity wave-induced time delay is of order $\Delta T \approx Lh\eta$, where h is the dimensionless amplitude of the gravitational wave. Therefore, waves of amplitude $h \gtrsim \eta$ would be expected to have drastic effects on lens time delays.

Based on this effect, Allen claimed that gravitational lenses could serve as gravitational wave detectors, [4,5] and that a bound could be placed on the amplitude of gravitational waves of cosmological wavelengths from the requirement that the wave-induced time delay in the double quasar 0957+561 not exceed the observed delay of 1.48 yr. [6] (The observed delay is generally attributed to the intrinsic delay of the lens.) For 0957+561, the image angular separation is 3×10^{-5} rad = 6.1 arcsec, and Allen obtained the bound $h < 2 \times 10^{-5}$. A gravity-wave background which nearly saturates the above bound would have important implications for gravitational lens models and would seriously compromise attempts to use lens time delays to measure the Hubble parameter.

Subsequent to Allen's work, the microwave anisotropy bound on the amplitude of cosmological gravitational waves has been significantly tightened by COBE. Through the Sachs-Wolfe effect, gravitational waves induce a temperature anisotropy of order their dimensionless amplitude. From the COBE detection of the quadrupole anisotropy, it follows that $h \lesssim (\delta T/T)_{\ell=2} \approx 6 \times 10^{-6}$ for wavelengths comparable to the present Hubble radius, $\lambda \sim H_0^{-1} = 3000$ h^{-1} Mpc. This bound is roughly a factor of three smaller than Allen's limit. However, although 0957+561 is the first gravitational lens system for which a time delay has been reliably measured, other lens systems are also being monitored; in particular, for a lens with smaller image angular separation η and thus smaller intrinsic time delay, the wave-induced delay would be even more important, and the corresponding lens bound on h potentially more restrictive. Inflationary models suggest that a significant fraction of the quadrupole anisotropy could be due to gravitational waves [3,7]. If this is the case, then time delays induced by gravitational waves in gravitationally lensed quasars would be significant.

In this paper, we reconsider Allen's proposal. Our central theme is that, for measurements of time delays in gravitational lenses to serve as gravitational wave detectors, the observer must be able to separate the wave-induced time delay from the intrinsic time delay originating in the lens geometry. We discuss the feasibility of observationally distinguishing these two sources of time delay. We approach this issue through application of Fermat's principle, a useful tool for analysing gravitational lens problems [8,9], which has recently

been shown to hold in the non-stationary space-times we consider [10,11]. We conclude that measurements of time delays in gravitational lenses are not likely to serve as a method to detect or constrain a cosmological background of gravitational waves, because the wave-induced time delay is observationally indistinguishable from the intrinsic time delay of an alternative lens geometry. As a consequence, the cosmological applications of lens time delays, e.g., inferring H_0 or galaxy masses, are not affected by gravity waves, regardless of the amplitude h. We note that, using quite different methods, the same conclusions were reached for general (scalar, vector, and tensor) metric perturbations by Frieman, Kaiser, and Turner. [12]

In Refs. [4,5], the time delay induced by a gravitational wave upon a gravitational lens was evaluated through the Sachs-Wolfe formula [1] for the differential photon redshift in the presence of metric fluctuations, integrated along unperturbed photon paths, *i.e.*, along the same trajectories the photons would have followed in the absence of the wave. As we will show, this method is not applicable in the case that the wave amplitude h is comparable to or larger than the angular separation 2η that the images would have in the absence of the wave, and the expression for the time delay derived in Refs. [4,5] is valid only if $h \ll \eta$. In the opposite limit, $h \gg \eta$, the effect of the wave is equivalent to a change in the alignment of the system so large that multiple images do not form (at least for non-singular lens potentials). Thus, the wave-induced time delays never exceed typical intrinsic delays, and cannot be used to constrain the amplitude of cosmological gravitational waves. Moreover, even in cases where the wave-induced delay is comparable to the typical intrinsic delay, we will show that an observer would attribute the entire delay to the intrinsic lens geometry. Thus, the wave-induced delay cannot be unearthed in practice or in principle.

To address these issues, we explicitly take into account the spatial distortion of the photon trajectories induced by the gravitational waves, which is non-negligible even if $h << \eta$. The wave-induced perturbation of the photon paths gives rise to extra contributions to the time delay, in addition to the differential redshift along the two trajectories. The extra contributions arise as a consequence of difference in path lengths and different gravitational

potential traversed by each photon due to the asymmetry in their trajectories induced by the wave. When the dust settles, our result for the wave-induced time delay coincides with that of Refs. [4,5] because these extra terms cancel each other, but only in the limits $h \ll \eta$ and $\omega L\eta \ll 1$. Moreover, the spatial distortion of the photon paths is always very significant when it comes to the interpretation of lens observations: a gravitational wave distorts the apparent angular positions of the images relative to the deflector in just such a way that an observer would attribute the wave-induced time delay to an intrinsic time delay associated with the image-deflector misalignment he or she sees. Since the lens geometry is not known *a priori*, but reconstructed from observations, one could equally well adjust the measurements to a given lens geometry in the presence of gravitational waves, or to an alternative lens geometry and no waves at all. Thus, it appears observationally impossible to distinguish wave-induced time delays from intrinsic delays, and so to detect cosmological gravitational waves through time delay measurements in gravitationally lensed quasars.

II. TIME DELAY IN A SIMPLE LENS CONFIGURATION

To more clearly display the features discussed above, we first analyze a simple lens model: a Schwarzschild (point mass) lens in a highly symmetric configuration, and a gravitational wave propagating perpendicular to the lens axis. In the next section we generalize the results derived here to the case of an arbitrary thin lens and arbitrary polarization and wave vector of the gravitational wave. Consider a static, spherical body of mass M, located at the origin of coordinates, that deflects photons emitted by a point-like source located at $(x = 0, y = 0, z = -L)$, and an observer on the extension of the source-deflector line at $(x = 0, y = 0, z = +L)$ (see Fig. 1). Given the axial symmetry, the observer sees an Einstein ring image of the source, with angular radius $\eta \equiv \sqrt{2GM/L}$. Here, G is Newton's constant, we take the speed of light $c = 1$, and we assume $\eta << 1$. To simplify the discussion, we focus on those photons that travel along the $y = 0$ plane, forming two images on opposite sides of the ring. In the absence of a gravitational wave, there is no time delay between the two

images, and they arrive with an angular separation $\Delta\theta = 2\eta$. Now consider a gravitational wave of dimensionless amplitude h and frequency ω, with polarization $(+)$, propagating along the x axis in the positive x direction. Sufficiently far from the deflector mass, the spacetime interval can be approximated by

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 + \frac{2GM}{r}\right)(dx^2 + dy^2 + dz^2) + h\cos\omega(t-x)(dy^2 - dz^2)\ . \quad (2.1)$$

Along a photon path, $ds^2 = 0$. Thus, if the spatial photon trajectories $x = x(z)$ were known, one could evaluate the time of travel by simple integration in z from $-L$ to L,

$$T \approx \int_{-L}^{L} dz \left[1 + \frac{1}{2}\left(\frac{dx}{dz}\right)^2 + \frac{1}{2}h\cos\omega(t-x) + \frac{2GM}{r}\right]\ . \quad (2.2)$$

To the level of approximation we shall be working (we are interested in terms of order $h\eta$ in the time delay), t can be replaced in eq. (2.2) by $t = t_e + (z + L)$, with t_e the time at which the photons were emitted at $(x = 0, z = -L)$. The first two terms in the integrand of eq. (2.2) are the geometric contribution to the time of travel, while the third and fourth are contributions from the gravitational potential of the wave and the deflector respectively.

A. Integration along unperturbed paths

Let us first evaluate the time delay along unperturbed photon trajectories. We approximate each path by straight segments, deflected by angle $\alpha = 2\eta$ at the deflector plane $z = 0$ (see Fig. 1). The approximation by straight segments is convenient and appropriate in the case of thin gravitational lenses, where most of the deflection occurs in the immediate vicinity of the deflector plane [9]. The light trajectories are then

$$\begin{aligned} x_{1,2} &= \pm\eta(z + L) \qquad z < 0 \\ x_{1,2} &= \mp\eta(z - L) \qquad z > 0 \end{aligned} \quad (2.3)$$

where the subscript (1,2) distinguishes trajectories that pass along opposite sides of the deflector. Straightforward integration leads to the time of travel, $T_{1,2}$. Clearly, given the

symmetry of the integration paths, the only contribution to the time delay comes from the different gravitational wave potential encountered by each trajectory, *i.e.*, from the third term in the integrand of eq. (2.2). The time delay is

$$\Delta T = T_1 - T_2 = -\frac{h\eta}{\omega}[\sin\omega(t_e + 2L) + \sin\omega t_e - 2\sin\omega(t_e + L)\cos\omega L\eta] \ , \tag{2.4}$$

which, if $\omega L\eta << 1$, is approximated by

$$\Delta T \approx 4\frac{h\eta}{\omega}\sin^2\left(\frac{\omega L}{2}\right)\sin\omega(t_e + L) \ . \tag{2.5}$$

This coincides with the result of Ref. [4], evaluated by integration of the Sachs-Wolfe formula along the same unperturbed trajectories.

The method used above to evaluate the time delay induced by the gravitational wave is questionable, even if $h << \eta$, because the actual photon trajectories are perturbed by the wave, and are expected to be neither straight nor symmetric with respect to the lens axis. As a result, one path may have smaller impact parameter with respect to the deflector than the other, and hence be deflected by a larger angle. This asymmetry in the paths leads to differences in both the geometric and potential contributions to the time of travel as large as that evaluated above. We shall evaluate these extra contributions, after derivation of the lens equation through Fermat's principle, and find that they cancel to leading order only if $h \ll \eta$. Thus, eq. (2.5) is only valid in this limit (moreover, in this limit, the wave-induced delay would be swamped by the intrinsic delay if the perfectly aligned symmetric lens were replaced by a misaligned lens such as that in Fig. 2). Furthermore, we will show that, even in this limit, the delay (2.5) would be attributed by the observer to intrinsic lens delay, that is, to an apparent misalignment between source, deflector, and observer.

To prepare for this result, we first briefly review the derivation of the lens equation and time delay for a misaligned lens in the absence of gravity waves.

B. Fermat's principle for a Schwarzschild lens

Fermat's principle provides a useful shortcut in many lensing problems: one can approximate the photon trajectories by null zig-zag trial paths and then extremize the time of travel [8,9], instead of solving the geodesic equations. Consider the lens depicted in Fig. 2. The source is at an angle β with respect to the line that joins observer and deflector, and in this subsection we assume there is no gravitational wave present. Consider a path that, starting from the source at $z = -L$, moves along a straight line up to the $z = 0$ plane, where it is deflected by an angle α, and then arrives at the observer at $z = L$, forming an angle θ with respect to the line that joins observer and deflector. The angles α, β and θ must satisfy

$$\alpha = 2(\theta - \beta) \ . \tag{2.6}$$

Assuming $\beta, \theta << 1$, the time of travel along such a null path would be

$$T \approx 2L + \theta^2 L - 2\beta\theta L - 4GM \ln\theta \ , \tag{2.7}$$

where we have neglected constant (θ-independent) terms. The first three terms are of purely geometric origin, while the last originates in the gravitational potential of the deflector. The condition that T be an extremum ($dT/d\theta = 0$ for fixed β) gives

$$\theta^2 - \beta\theta - \frac{2GM}{L} = 0 \ , \tag{2.8}$$

which is usually referred to as the lens equation [9]. The solutions give the angular positions of the two images:

$$\theta_{1,2} = \frac{1}{2}(\beta \pm \sqrt{\beta^2 + 4\eta^2}) \qquad \text{with} \qquad \eta^2 \equiv \frac{2GM}{L} \ . \tag{2.9}$$

Different signs indicate that the images appear on opposite sides of the deflector. The resulting time delay is given by

$$\Delta T = T_1 - T_2 = (\theta_1^2 - \theta_2^2)L - 2\beta(\theta_1 - \theta_2)L - 4GM \ln(\theta_1/|\theta_2|) \ . \tag{2.10}$$

In the limit of small misalignment angle, $\beta \ll \eta$, the image angular positions are approximately

$$\theta_{1,2} \approx \pm\eta + \frac{1}{2}\beta \ . \tag{2.11}$$

and the time delay reduces to

$$\Delta T \approx -4\beta\eta L \approx -2\beta\Delta\theta \ , \tag{2.12}$$

where $\Delta\theta = \theta_1 - \theta_2$ is the image angular separation.

For the arguments we will make below, it is useful to bear in mind how lens observations in such a simple system could be used to extract cosmological information. If the deflector is seen in addition to the two images, then the lens observables are θ_1, θ_2, and ΔT. The observer can then infer the misalignment angle from $\beta = \theta_1 + \theta_2$, and the lens parameter η from eq. (2.9). Using these observed and derived quantities, eq. (2.10) can be used to determine L. (More generally, if the source and observer are not equidistant from the lens, the reasoning above determines a distance measure for the lens.) Comparison with the deflector redshift then yields an estimate of the Hubble parameter H_0. For deflectors more complex than point masses, the observables above must be supplemented by information about the deflector potential obtained, e.g., from measurement of the lens' velocity dispersion.

C. A Schwarzschild lens with a gravitational wave

Now we proceed with a similar technique, based on Fermat's principle, to evaluate the time delay induced by a gravitational wave in the symmetric Schwarzschild lens configuration, with source, deflector, and observer aligned as in Fig. 1. The validity of Fermat's principle in non-stationary spacetimes was recently discussed in the context of gravitational lensing problems [10,11]. We evaluate the time of travel, integrating eq. (2.2). Instead of integrating along straight lines, however, we take the two segments of each zig-zag trial path to be null geodesics of the gravitational wave metric, as they would have been in the absence of the lensing body. We work up to the order of approximation needed to study terms of order $h\eta$ in the time delay, and we also assume $\omega L\eta << 1$. This is the most interesting

range, since the effect under consideration becomes largest when $\omega L \approx 1$. The geodesic equations in the metric (2.1), with $M = 0$, lead to

$$\frac{dx}{dz} \approx \gamma - \frac{1}{2} h \cos \omega (t_e + z + L) \ . \tag{2.13}$$

Here, γ is an arbitrary integration constant, the average slope of the trajectory, assumed to be small. Note that, as before, in the argument of the cosine in (2.13), we have replaced t by $t_e + z + L$; we have also dropped the dependence on x because it is unnecessary to include it to evaluate the time delay to order $h\eta$, in the limit $\omega L \eta << 1$. The third term in eq. (2.2) is the only one where the x-dependence inside the cosine needs to be included, and it is enough to do so at zero order.

Now we choose the integration constants so that a trajectory that starts from $x = 0, z = -L$ at $t = t_e$, and deflected by an arbitrary angle in the $z = 0$ plane, arrives at the observer at $x = 0, z = L$. One finds that

$$\begin{aligned} \frac{dx}{dz} &= \ \epsilon - \frac{1}{2} h \cos \omega (t_e + z + L) \quad \text{if} \quad z < 0 \\ \frac{dx}{dz} &= -\epsilon + \frac{h}{2\omega L} [\sin \omega (t_e + 2L) - \sin \omega t_e] - \frac{1}{2} h \cos \omega (t_e + z + L) \quad \text{if} \quad z > 0 \ . \end{aligned} \tag{2.14}$$

Here, ϵ is an integration constant which parametrizes the family of trajectories that meet the focusing conditions at the required points. Each trajectory consists of two segments which are null geodesics of the gravitational wave metric, neglecting the deflector potential, which will be taken into account through Fermat's principle. According to the latter, the actual trajectories are those null paths that extremize the time of travel with respect to variations of the parameter ϵ in the metric that includes both the gravitational wave as well as the deflector's potential.

Before we proceed to extremize, however, we parametrize the trajectories in a different way, defining a parameter more relevant to observations. Instead of ϵ, we use the angular position of the image, which we denote by θ, relative to the angular position of the deflector, at the time of arrival of the images at the observer. We again use equation (2.13) and fix the appropriate integration constants to determine the slope of the trajectory of a photon

that arrives from the deflector (i.e., the angular position of the deflector), which we denote by $dx_{\rm lens}/dz$,

$$\left.\frac{dx_{\rm lens}}{dz}\right|_{z=L} = \frac{h}{2\omega L}[\sin\omega(t_e+2L) - \sin\omega(t_e+L)] - \frac{1}{2}h\cos\omega(t_e+2L)\ . \tag{2.15}$$

Then the angular position θ of the image with respect to the apparent deflector position is

$$\theta = -\left.\frac{dx}{dz}\right|_{z=L} + \left.\frac{dx_{\rm lens}}{dz}\right|_{z=L} = \epsilon + \frac{h}{2\omega L}[\sin\omega t_e - \sin\omega(t_e+L)]\ . \tag{2.16}$$

This relates θ to the parameter ϵ of eq. (2.14). Now the deflection imprinted by the lens upon the trajectory at $z=0$, which we denote by α, can be written as

$$\alpha \equiv \left.\frac{dx}{dz}\right|_{z=0^-} - \left.\frac{dx}{dz}\right|_{z=0^+} = 2(\theta-\beta_g) \tag{2.17}$$

where we have defined

$$\beta_g \equiv \frac{h}{4\omega L}[\sin\omega(t_e+2L) + \sin\omega t_e - 2\sin\omega(t_e+L)] = -\frac{h}{\omega L}\sin^2\left(\frac{\omega L}{2}\right)\sin\omega(t_e+L)\ . \tag{2.18}$$

We have defined the quantity β_g in such a way that the expression (2.17) for the deflection has the same form as eq. (2.6)–in that case, β measured the misalignment between deflector, source, and observer in the absence of a gravitational wave, as in Fig. 2. We will see in what follows that in all respects β_g plays exactly the same effective role here.

Next we evaluate the time of travel, integrating eq. (2.2) along the null trajectories (2.14), parametrized in terms of θ, and find

$$T \approx 2L + \theta^2 L - 2\beta_g\theta L - 4GM\ln\theta\ . \tag{2.19}$$

As advertised, this has exactly the form of eq. (2.7), which gave the time of travel for a similar lens with no gravitational wave, but with lens and source misaligned by an angle β, as in Fig. 2. Recall that in that case, the first three terms were of geometric origin. In the present case, only the first two terms are geometric (they come from integration of $[1+(dx/dz)^2/2]$ in (2.2)). The third term, proportional to β_g, is due to the wave gravitational

potential, and comes from integration of the third term in eq. (2.2). Finally, the last term is due to the deflector's gravitational potential.

The equivalence of expressions (2.7) and (2.19) leads to our main conclusion: the lens equation in the presence of a gravitational wave is, to the order of approximation considered, completely equivalent to that of a similar lens with a different alignment and no gravity wave. The effective misalignment angle β_g is given by eq. (2.18) in terms of the wave parameters. The analogy is exact only in the limit $\omega L\eta \ll 1$, but this is the interesting range in any case, because $\eta \ll 1$ and time delays are largest for $\omega L \approx 1$. Since β_g depends upon time, the analogy is only valid over periods of time much shorter than ω^{-1}. Again, since the effect is relevant only for waves of cosmological wavelengths, this time-variation of the time delay is observationally irrelevant.

From eq. (2.19), the time delay between the two images is given by expression (2.10) with the substitution $\beta \to \beta_g$,

$$\Delta T = T_1 - T_2 = (\theta_1^2 - \theta_2^2)L - 2\beta_g(\theta_1 - \theta_2)L - 4GM\ln(\theta_1/|\theta_2|) \ . \tag{2.20}$$

So far, we have made no assumption about the relative amplitudes of the gravitational wave effect, $\beta_g \sim h$, and the deflector Einstein ring angular radius η (Cf. eq. (2.9)). However, if $\beta_g \gg \eta$, the effect of the wave is equivalent to that of a system very much out of alignment. In this limit, the magnification of the second image goes to zero as $(\eta/\beta_g)^4$, and multiple image formation effectively does not take place.

In the opposite limit, $\beta_g \ll \eta$, Fermat's principle leads to the same result as eq. (2.12), but with β replaced by the effective β_g of eq. (2.18),

$$\Delta T \approx -4\beta_g\eta L = 4\frac{h\eta}{\omega}\sin^2\left(\frac{\omega L}{2}\right)\sin\omega(t_e + L) \ . \tag{2.21}$$

This result coincides with that of eq. (2.5), obtained through integration along unperturbed paths, and is just Allen's result [4]. Note that the additional term originating in a path length difference, $(\theta_1^2 - \theta_2^2)L$, cancels the term due to the deflector's gravitational potential, $4GM\ln(\theta_1/|\theta_2|)$. The wave-induced distortion of the photon paths is not negligible, however, when it comes to interpreting the result.

Indeed, suppose the observer of this lens has no knowledge of the possible existence of gravitational waves, and seeks to measure, e.g., the deflector mass M or the Hubble parameter H_0 from her observations. The effect of the gravity wave upon the apparent angular positions of the images and the deflector would trick the observer into believing he or she sees an ordinary misaligned lens. Moreover, the observer's inference of the misalignment angle β_g from the observed image angular positions, $\beta_g = \theta_1 + \theta_2$, and the observed image time delay would all be in accord with this belief, and he or she will infer the *correct* values for M and H_0, even though taking no account of gravitational waves and instead assuming a homogeneous and isotropic spacetime (aside from the deflector). That is, while the gravity wave does cause a time delay, it covers its own tracks in a misalignment change, leaving no measurable trace of its presence, and can be safely and consistently ignored by the lens observer. Thus it appears impossible to use time delay measurements to detect cosmological gravitational waves even in principle.

For completeness, we emphasize that this conclusion holds even if $\beta \gtrsim \eta$, but that it is only in the limit $\beta_g \ll \eta$ that the time delay agrees with eq. (2.5).

III. TIME DELAY IN A THIN LENS WITH GRAVITATIONAL WAVES

In this section we show how the conclusions reached above can be generalized to the case of an arbitrary thin gravitational lens with gravitational waves of arbitrary polarization and direction of propagation. First we briefly review the features of a general lens when no gravitational waves are present. We assume, as usually applies for cases of astrophysical interest, a thin, stationary gravitational lens, such that the weak field approximation is valid [9]. Consider a lens geometry as in Fig. 2, only now we do not assume that the photon paths lie in the plane that contains source, deflector, and observer: $\vec{\theta}$ and $\vec{\beta}$ are two-component angular vectors, that give angular positions at the observer's location, $\vec{\alpha}$ is the deflection, and $\vec{\xi} = L\vec{\theta}$ determines the impact parameter in the deflector plane. The condition that the photons from the source reach the observer implies the following relation, the vectorial

generalization of eq. (2.6):

$$\vec{\alpha} = 2\ (\vec{\theta} - \vec{\beta})\ , \tag{3.1}$$

and the time of travel is given by:

$$T \approx 2L + |\vec{\theta}|^2\ L - 2\ \vec{\beta} \cdot \vec{\theta}\ L - \psi(\vec{\xi})\ , \tag{3.2}$$

which is the generalization of eq. (2.7). The last term originates in the deflector gravitational potential, and is given, for a thin lens, by

$$\psi(\vec{\xi}) = 4G \int d^2\xi'\ \Sigma(\vec{\xi'}) \ln\left(\frac{|\vec{\xi} - \vec{\xi'}|}{\xi_0}\right)\ , \tag{3.3}$$

where Σ is the mass density projected on the lens plane. Note that this term depends only upon the impact parameter $\vec{\xi}$, reflecting the fact that in the thin lens approximation the effect of the gravitational potential of the lens is dominated by that part of the trajectory closest to the deflector. Here ξ_0 is an arbitrary length scale. Following Fermat's principle, we extremize the time of travel with respect to $\vec{\theta}$ and arrive at the lens equation:

$$\vec{\theta} - \vec{\beta} - \frac{1}{2L}\ \frac{\partial\psi}{\partial\vec{\theta}} = 0\ . \tag{3.4}$$

Notice that $\partial\psi/\partial\vec{\theta} = L\partial\psi/\partial\vec{\xi}$. The solutions to this equation give the angular positions of the images.

Now we show the equivalence between the effect of a gravitational wave and an effective lens misalignment. Consider a lens geometry with deflector, source, and observer aligned at $z = 0, -L, L$ respectively, as in Fig. 1, along the z-axis. Let U be the gravitational potential of the deflector. Consider a gravitational wave propagating at an angle ϑ with respect to the lens axis. We take the (x, z) plane as that containing the lens axis and the direction of propagation of the gravitational wave. The metric perturbation caused by the wave can be expressed as:

$$h_{ij} = \begin{pmatrix} -\cos^2\vartheta\ h_+ & -\cos\vartheta\ h_\times & \sin\vartheta\cos\vartheta\ h_+ \\ -\cos\vartheta\ h_\times & h_+ & \sin\vartheta\ h_\times \\ \sin\vartheta\cos\vartheta\ h_+ & \sin\vartheta\ h_\times & -\sin^2\vartheta\ h_+ \end{pmatrix} \cos(\omega t - \vec{k}\cdot\vec{x}) \tag{3.5}$$

with propagation vector $\vec{k} = \omega(\sin\vartheta, 0, \cos\vartheta)$, and h_+ and $h_\times$ the amplitudes of the two wave polarizations. The total metric is then given by:

$$ds^2 = (1+2U)dt^2 - (1-2U)(dx^2+dy^2+dz^2) + h_{ij}dx^i dx^j \ . \tag{3.6}$$

Along a null path, the time of travel is given by

$$T \approx \int_{-L}^{L} dz \left[1 + \frac{1}{2}\left(\frac{dx}{dz}\right)^2 + \frac{1}{2}\left(\frac{dy}{dz}\right)^2 + \frac{1}{2}h_{ij}\frac{dx^i}{dz}\frac{dx^j}{dz} - 2U\right] \ . \tag{3.7}$$

Notice that $\int 2U\ dz$ is the same as what we had previously defined by ψ in eq. (3.2).

We now define a family of null trial paths along which we will integrate eq. (3.7). Each path is built out of two segments deflected by an angle $\vec{\alpha}$ at the deflector plane. Instead of taking straight trajectories, we let each segment be a solution of the geodesic equations in the presence of the gravitational wave, neglecting the potential U of the deflector, since its effects are later taken into account through Fermat's principle. The condition that a photon from the source at $(x, y = 0, z = -L)$ reaches the observer at $(x, y = 0, z = L)$ defines a one-parameter family of trajectories parametrized by an arbitrary vector $\vec{\epsilon} = (\epsilon_x, \epsilon_y)$:

$$\begin{aligned}
\frac{dx}{dz} &= \epsilon_x - \frac{h_+}{2}\sin\vartheta(1-\cos\vartheta)\cos\omega(t_e + L + z(1-\cos\vartheta)) \\
\frac{dy}{dz} &= \epsilon_y + h_\times \sin\vartheta\cos\omega(t_e + L + z(1-\cos\vartheta)) \qquad \text{if} \quad z < 0 \quad , \\
\frac{dx}{dz} &= -\epsilon_x + \frac{h_+}{2\omega L}\sin\vartheta[\sin\omega(t_e + L(2-\cos\vartheta)) - \sin\omega(t_e + L\cos\vartheta)] \\
&\quad - \frac{h_+}{2}\sin\vartheta(1-\cos\vartheta)\cos\omega(t_e + L + z(1-\cos\vartheta)) \\
\frac{dy}{dz} &= -\epsilon_y - \frac{h_\times}{\omega L}\frac{\sin\vartheta}{(1-\cos\vartheta)}[\sin\omega(t_e + L(2-\cos\vartheta)) - \sin\omega(t_e + L\cos\vartheta)] \\
&\quad + h_\times \sin\vartheta\cos\omega(t_e + L + z(1-\cos\vartheta)) \qquad \text{if} \quad z > 0 \ .
\end{aligned} \tag{3.8}$$

The wave also affects the apparent position of the deflector; as before, we change variables from $\vec{\epsilon}$ to the relative angular position between the image and the deflector at the observer's position, which we denote by $\vec{\theta}$. We find the relation (assuming $\omega L\theta << 1$):

$$\vec{\theta} = (\epsilon_x, \epsilon_y) + \left(\frac{h_+ \ \sin\vartheta}{2\omega L}, \frac{-h_\times \ \sin\vartheta}{\omega L(1-\cos\vartheta)}\right)[\sin\omega(t_e + L\cos\vartheta) - \sin\omega(t_e + L)] \ . \tag{3.9}$$

The deflection $\vec{\alpha}$ imprinted upon the trajectory at the lens plane $z = 0$ can be written as

$$\vec{\alpha} = 2(\vec{\theta} - \vec{\beta}_g) \tag{3.10}$$

if we define $\vec{\beta}_g$ as

$$\begin{aligned}\vec{\beta}_g \equiv & \left(\frac{h_+}{4\omega L}\sin\vartheta, \frac{-h_\times}{2\omega L}\frac{\sin\vartheta}{(1-\cos\vartheta)}\right) \\ & \times[\sin\omega(t_e + L(2-\cos\vartheta)) + \sin\omega(t_e + L\cos\vartheta) - 2\sin\omega(t_e + L)] \\ = & -\left(\frac{h_+}{\omega L}\sin\vartheta, \frac{-h_\times}{\omega L}\frac{2\sin\vartheta}{(1-\cos\vartheta)}\right)\sin^2\left[\frac{\omega L}{2}(1-\cos\vartheta)\right]\sin\omega(t_e + L)\end{aligned} \tag{3.11}$$

Of course, eqs. (3.9) and (3.11) reduce to our previous eqs. (2.16) and (2.18) in the case $\vartheta = \pi/2, h_\times = 0$.

Now we are ready to find the time of travel, integrating eq. (3.7). One important thing to note is that, for a thin lens, integration of the last term in (3.7), the contribution of the deflector gravitational potential U, gives $-\psi(\vec{\xi})$, where $\vec{\xi} = \vec{x}(z = 0)$ is the impact parameter of the trajectory. And, if $\omega L\theta << 1$, the relation $\vec{\xi} = L\vec{\theta}$ still holds, as in the absence of a gravitational wave. At the end, we find for the total time of travel:

$$T = 2L + |\vec{\theta}|^2\ L - 2\ \vec{\beta}_g \cdot \vec{\theta}\ L - \psi(L\vec{\theta})\ , \tag{3.12}$$

Since this has exactly the same functional dependence on $\vec{\theta}$ as in eq. (3.2), we confirm the equivalence between an aligned lens in the presence of a gravitational wave and a lens with an effective lack of alignment given in terms of the wave parameters by $\vec{\beta}_g$ of eq. (3.11).

In the special case of an axially symmetric lens and in the limit $|\vec{\beta}| << |\vec{\theta}|$, the solutions $\vec{\theta}_{1,2}$ to the lens equation lie in the plane that contains the lens axis and the direction of $\vec{\beta}_g$. That plane forms an angle ϕ with the plane that contains the gravitational wave propagation vector (which we took as the (x, z) plane) such that

$$\tan\phi = \frac{(\beta_g)_y}{(\beta_g)_x} = -\frac{2h_\times}{h_+(1-\cos\vartheta)}\ . \tag{3.13}$$

The solutions are then of the form

$$\vec{\theta}_{1,2} \approx \pm\vec{\eta} + a\vec{\beta}\ , \tag{3.14}$$

with

$$\vec{\eta} = \eta(\cos\phi, \sin\phi) \quad ; \quad \eta = \frac{1}{2L}\frac{\partial\psi}{\partial\theta}\Big|_{\theta=\eta} \tag{3.15}$$

a solution to the unperturbed lens equation, and a a coefficient that depends upon the lens model:

$$a^{-1} = 1 - \frac{1}{2L}\frac{\partial^2\psi}{\partial\theta}\Big|_{\theta=\eta} \ . \tag{3.16}$$

For instance, $a = 1/2$ for a Schwarzschild lens, where $\psi = 4GM\ln\theta$, and $a = 1$ for a singular isothermal sphere, where $\psi = 4\pi\Sigma_v^2\theta$, with Σ_v the velocity dispersion. For a non-singular lens, there must be an odd number of images; in that case, eq. (3.14) refers, say, to the outer two images (the central third image is usually de-magnified).

The time of travel for these solutions can be expanded as

$$T_{1,2} = 2L + (|\vec{\eta}|^2 \pm 2a\vec{\beta}\cdot\vec{\eta})L \mp 2\vec{\beta}\cdot\vec{\eta}L - \psi(\pm L\vec{\eta}) - a\vec{\beta}\cdot\frac{\partial\psi}{\partial\vec{\theta}}\Big|_{\vec{\theta}=\pm\vec{\eta}} \ . \tag{3.17}$$

Using the unperturbed lens equation (3.15) we see that the two contributions to the time of travel proportional to $a\vec{\beta}$, one of geometric origin and the other due to the deflector's gravitational potential, cancel each other. Besides, $\psi(L\vec{\eta}) = \psi(-L\vec{\eta})$ for an axially symmetric lens. Finally, the time delay between two images for a thin, axially symmetric lens, in the limit $|\vec{\beta}| << |\vec{\eta}|$ and with $\omega L\eta << 1$ is

$$\Delta T \approx -4\vec{\beta}\cdot\vec{\eta}L = \frac{4\eta}{\omega}\sin\vartheta\left[h_+\cos\phi + h_\times\frac{2\sin\phi}{(1-\cos\vartheta)}\right]\sin^2\left[\frac{\omega L}{2}(1-\cos\vartheta)\right]\sin\omega(t_e+L) \ , \tag{3.18}$$

where ϕ is the angle between the plane containing the photon trajectories and the plane that contains the gravitational wave propagation vector, as given by (3.13). In the limit $\beta_g << \eta$, expression (3.18) agrees with that of Ref. [4], evaluated through integration of the Sachs-Wolfe formula along unperturbed photon paths.

IV. CONCLUSIONS

We have shown that the lens equation for a thin, axially aligned gravitational lens configuration in the presence of a very long wavelength gravitational wave is equivalent to that of a similar lens with the source out of alignment and no gravitational wave. An observer who measures time delays, angular positions, or any other observables such as relative magnifications and redshifts, and uses them to reconstruct the lens configuration, cannot tell the two situations apart. Thus, an observer ignorant of gravitational waves would naturally and 'correctly' interpret the observations as a simple non-aligned lens. This conclusion is valid if $\omega L\eta << 1$, which is the interesting range since the induced time delays are largest when $\omega L \approx 1$. We performed our calculations around an aligned lens configuration and with source and observer equidistant from the deflector, but it is clear that the conclusion holds in more general cases: the effect of a long wavelength gravitational wave upon a lens with a given geometry is equivalent, from the observer's viewpoint, to an effective change of lens geometry. Consequently, measuring time delays in gravitational lenses does not provide a method for probing a cosmological background of gravitational waves.

Formally, Allen's result for the wave-induced time delay is correct in the small amplitude limit: for $\beta_g << \eta$, eq. (3.18) for the time delay induced by a gravitational wave upon a thin, axially symmetric lens agrees with that of Ref. [4]. In the opposite limit, $\beta_g \gg \eta$, the effect of the gravitational wave is equivalent to a change in the alignment between source, deflector, and observer by an amount that exceeds the typical deflection angle the deflector can imprint, precluding the formation of multiple images in the case of an aligned lens. In this case, multiple images can only form if there is a compensating geometric misalignment between source and deflector, and the geometric delay will partially cancel the lens-induced delay. Thus, even if $h >> \eta$, the total time delay does not exceed the typical intrinsic lens time delay of order $L\eta^2$. Moreover, in either limit, the measured time delay is just what the observer would expect in the complete absence of gravitational waves, based on her measurements of the lens observables.

The detection by the COBE satellite [2] of a quadrupole anisotropy in the cosmic microwave background places a bound $h \lesssim 6 \times 10^{-6}$ on the amplitude of cosmological gravitational waves. If a large fraction of the anisotropy detected by COBE is due to gravitational waves, a possibility that can be accommodated by many inflationary cosmological models [7,3], then the wave-induced time delays between multiple images of quasars are comparable to typical intrinsic lens time delays, with $\eta \approx 10^{-5}$. One could have hoped that careful lens modelling could allow one, at least in principle if not in practice, to separate the wave-induced from the intrinsic time delay, and thus reveal the presence of the gravitational waves. Our work indicates that this is not a possibility.

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REFERENCES

[1] R. Sachs and A. Wolfe, Astrophys. J. **147**, 173 (1967).

[2] G.F. Smoot etal., Astrophys. J. **396**, L1 (1992).

[3] L. Krauss and M. White, Phys. Rev. Lett. **69**, 869 (1992); R.L. Davis, H.M. Hodges, G.F. Smoot, P.J. Steinhardt and M.S. Turner, Phys. Rev. Lett. **69**, 1856 (1992); F. Lucchin, S. Matarrese and S. Mollerach, Astrophys. J. **401**, L49 (1992); D. Salopek, Phys. Rev. Lett. **69**, 3602 (1992); T. Souradeep and V. Sahni, Mod. Phys. Lett. **A7**, 3531 (1992); R. Crittenden, et al., Phys. Rev. Lett. **71**, 324 (1993).

[4] B. Allen, Phys. Rev. Lett. **63**, 2017 (1989).

[5] B. Allen, Gen. Relativ. Grav. **22**, 1447 (1990).

[6] J. Lehar, J. N. Hewitt, D. H. Roberts, and B. F. Burke, Astrophys. J. **384** 453 (1992); W.H. Press, G.B. Rybicki and J.N. Hewitt, Astrophys. J. **385**, 404 (1992); ibid., 416 (1992). This is somewhat larger than the earlier time delay estimate of 415 days by C. Vanderriest et al., Astron. Astrophys. **215**, 1 (1989).

[7] L. Abbott and M. Wise, Nucl. Phys. B **244**, 541 (1984).

[8] R. Blandford and R. Narayan, Astrophys. J. **310**, 569 (1986).

[9] P. Schneider, J. Ehlers and E.E. Falco, *Gravitational Lenses*, Springer-Verlag, 1992.

[10] I. Kovner, Astrophys. J. **351**, 114 (1990).

[11] R. Nityananda and J. Samuel, Phys. Rev. D **45**, 3862 (1992).

[12] J. Frieman, N. Kaiser, and M. S. Turner, unpublished (1990), and in preparation (1994).

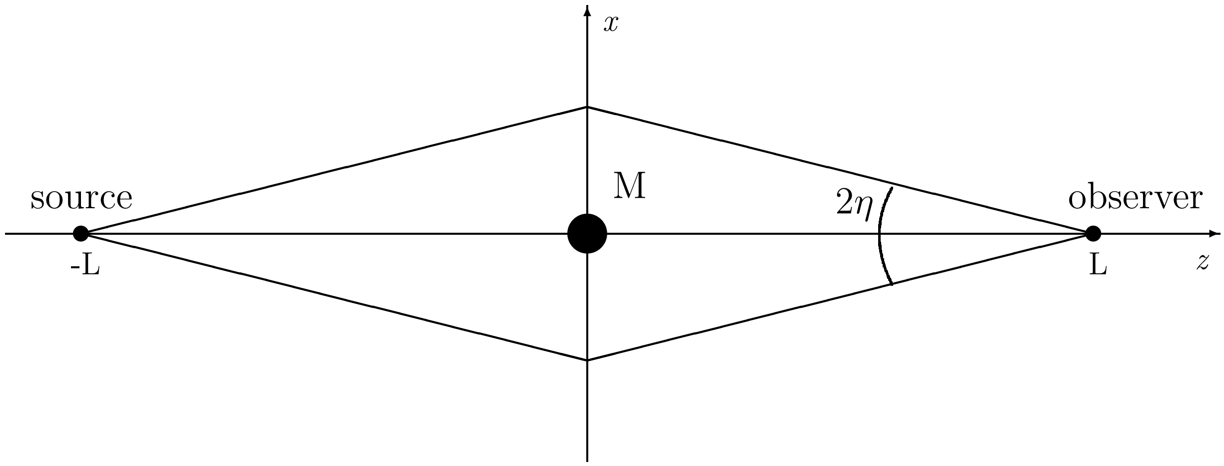

Figure 1: The geometry of an aligned lens. The deflector is a point mass M at the origin of coordinates. Source and observer lie along the z-axis, equidistant from the deflector. The observer sees two images of the same source (actually an Einstein ring) with angular separation 2η. The trajectories are approximated by straight segments.

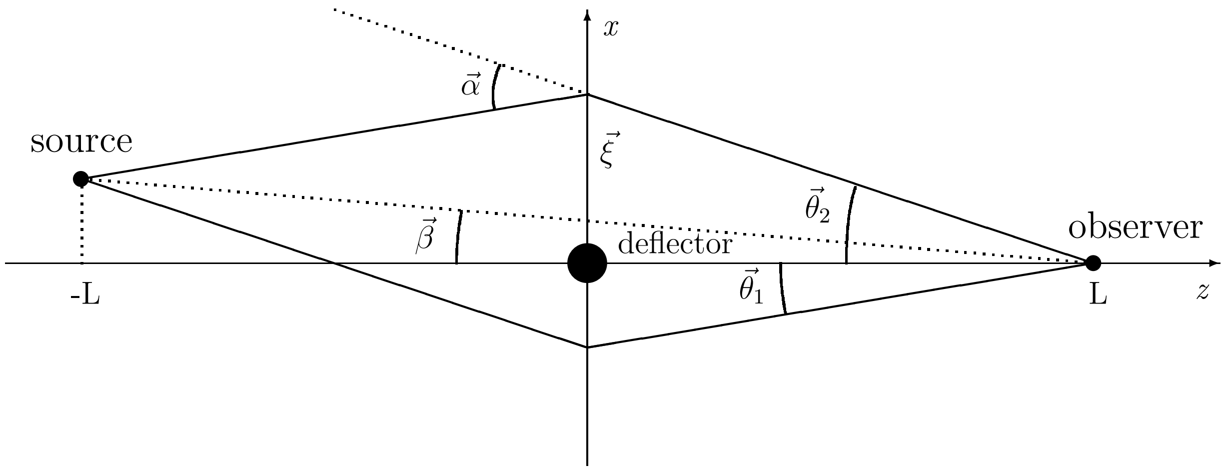

Figure 2: The geometry of a non-aligned lens. The source forms an angle $\vec{\beta}$ with respect to the line that joins observer and deflector. $\vec{\theta}_1$ and $\vec{\theta}_2$ are the angular positions of the images, $\vec{\xi}$ is the impact parameter, and $\vec{\alpha}$ the deflection angle.